Induction and secant varieties to Chow varieties

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Chow varieties

Let V be a finite dimensional $\mathbb{C}$-vector space. For a fixed d, consider the map

$$\mathrm{Ch}_d : \mathbb{P}(V)^{\times d} \to \mathbb{P}(\mathrm{Sym}^d V)$$
$$([v_1], \ldots, [v_d]) \to [v_1 \cdots v_d]$$

The image of this map, which we denote $\mathrm{Ch}_d(\mathbb{P}^n)$ if $\dim V = n + 1$, is a *Chow variety* (of 0-cycles), *split variety*, or *variety of complete decomposable (or reducible) forms*.

Question

For a given n, d, s, what is $\dim \sigma_s(\mathrm{Ch}_d(\mathbb{P}^n))$?

Chow rank

For generic $f \in \sigma_s(\mathrm{Ch}_d(\mathbb{P}^n))$,

$$f = \sum_{i=1}^{s} \ell_{i,1} \cdots \ell_{i,d}.$$

Computational complexity: This decomposition gives us an efficient way of evaluating f.

Expected dimension

$$\operatorname{expdim} \sigma_s(\mathrm{Ch}_d(\mathbb{P}^n)) = \min \left\{ s(dn+1), \binom{n+d}{d} \right\} - 1$$

$s(dn+1) \leq \binom{n+d}{d} \implies$ subabundant

$s(dn+1) \geq \binom{n+d}{d} \implies$ superabundant

Defective cases

Theorem

If $n \geq 4$ *and* $2 \leq s \leq \frac{n}{2}$*, then* $\sigma_s(\mathrm{Ch}_2(\mathbb{P}^n))$ *is defective.*

Proof.

Note that $\mathrm{Ch}_2(\mathbb{P}^n) = \tau(\nu_2(\mathbb{P}^n))$, the tangential variety of the Veronese variety of quadrics. The defective cases for its secant varieties were identified in [CGG02]. □

Conjecture

With the exception of the known defective cases, $\sigma_s(\mathrm{Ch}_d(\mathbb{P}^n))$ is always nondefective.

Known nondefective cases

Theorem

For the following n, d, s, $\sigma_s(\mathrm{Ch}_d(\mathbb{P}^n))$ *is nondefective.*

(a) $n = 1$, $s = 2$, $d \geq 3$ *and* $s \leq 2\left\lfloor\frac{n+1}{3}\right\rfloor$ *[CCGO17]*

(b) $s \geq \binom{n+d-1}{n}$ *[CGG$^+$19]*

The above improved on some earlier results [AB11, Shi12].

Terracini's lemma

Lemma

For any n, d, s,

$$\dim \sigma_s(\mathrm{Ch}_d(\mathbb{P}^n)) = \dim \sum_{i=1}^{s} \sum_{j=1}^{d} \ell_{i,1} \cdots \ell_{i,j-1} \ell_{i,j+1} \cdots \ell_{i,d} V - 1$$

for generic $\ell_{i,j} \in V$.

Abo-Ottaviani-Peterson induction

In [AOP09], an induction technique was developed for proving the nondefectivity of many cases of $\sigma_s(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$.

Basic idea of induction

What happens if we specialize some of the linear forms from Terracini's lemma to be the basis element x_0 of V, and the others so that they belong to $\langle x_1, \ldots, x_n \rangle$?

Then we may decompose the Terracini space into a direct sum:

$$W_1 \oplus x_0 W_2 \oplus x_0^2 W_3$$

If each piece has the expected dimension (in $(n-1, d)$, $(n-1, d-1)$, and $(n-1, d-2)$, resp.), then the entire space has the expected dimension.

Drawback: Only works in subabundant case.

Induction on n

Theorem ([Tor17])

If

$$\dim \sigma_s(\mathrm{Ch}_d(\mathbb{P}^{n_0})) = s(dn_0 + 1) - 1,$$

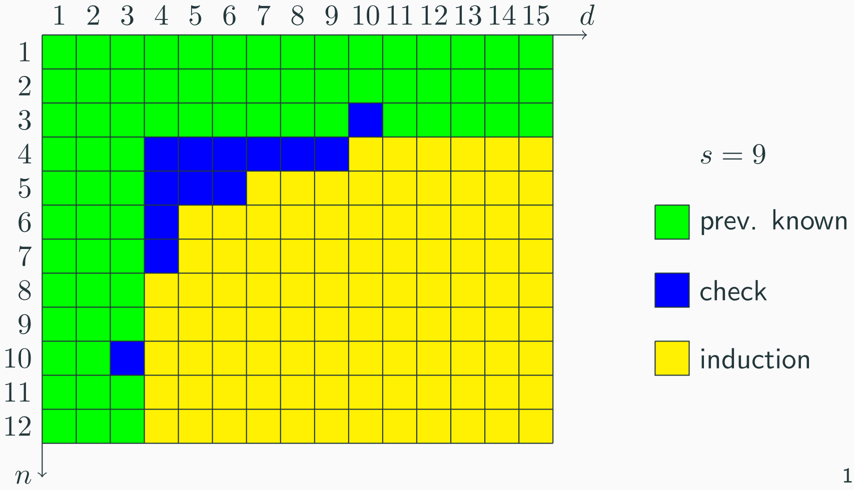
then

$$\dim \sigma_s(\mathrm{Ch}_d(\mathbb{P}^{n})) = s(dn + 1) - 1$$

for all $n \geq n_0$.

Computations

For fixed s, this reduces finding $\dim \sigma_s(\mathrm{Ch}_d(\mathbb{P}^n))$ for all n, d to checking finitely many base cases.



Small s

Theorem ([Tor17])

If $s \leq 35$, then

$$\dim \sigma_s(\mathrm{Ch}_d(\mathbb{P}^n)) = \min\left\{ s(dn+1), \binom{n+d}{d} \right\} - 1,$$

except for the previously known defective cases.

Brambilla-Ottaviani induction

In [BO08], there is a new proof for the nondefectivity of $\sigma_s(\nu_3(\mathbb{P}^n))$ ($n \neq 4$).

The s points are specialized onto up to 3 subspaces and induction with a step size of 3 is used.

How it works

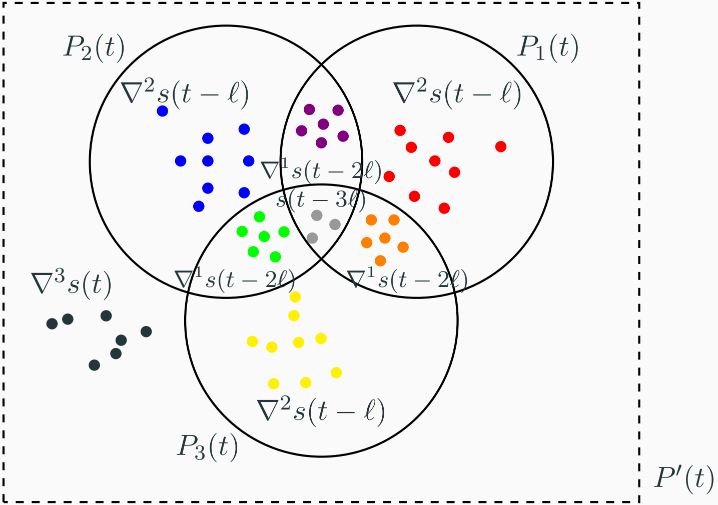
Theorem (Newton backward difference formula)

If ∇ is the backward difference operator with step size ℓ, i.e., $\nabla^0 s(t) = s(t)$ and $\nabla^i s(t) = \nabla^{i-1} s(t) - \nabla^{i-1} s(t-\ell)$, then

$$s(t) = \sum_{j=0}^{K} \binom{K}{j} \nabla^{K-j} s(t - j\ell).$$

If we have K subspaces, then specialize $\nabla^{K-j} s(t - j\ell)$ points onto each intersection of j of them.

Visualization



Quasipolynomials

If $s(t) = s_r(t) \in \mathbb{Q}[t]$ when $t \equiv r \pmod{\ell}$, $r = 0, \ldots, \ell - 1$, then s is *quasipolynomial* with period ℓ.

Furthermore:

(a) If s is *quasipolynomial* with period ℓ, then $\nabla^{K-j} s(t - j\ell)$ will be nice.

(b) $s(n) = \left\lfloor \frac{\binom{n+d}{d}}{dn+1} \right\rfloor$ (fixing d) and $s(d) = \left\lfloor \frac{\binom{n+d}{d}}{dn+1} \right\rfloor$ (fixing n) are quasipolynomial

Induction

If we have the expected dimension with

(a) i subspaces and $n - \ell + 1$ variables (or degree $d - \ell$) and

(b) $i + 1$ subspaces and $n + 1$ variables (or degree d),

then we have the expected dimension with i subspaces and $n + 1$ variables (or degree d)

Furthermore, if the degree of the quasipolynomial s is $K - 1$, and we have the expected dimension with K subspaces and $n + 1$ variables (or degree d), then we have the expected dimension with more variables (or higher degree).

Just compute finitely many base cases (up to $n \leq K\ell + 1$ or $d \leq K\ell + 1$), both subabundant and superabundant.

Results

(a) $n = 2$ ($\ell = 4$) [Abo14]

(b) $n, d = 3$ (gap, $\ell = 6$) [Abo14]

(c) $n = 3$ (slightly smaller gap, $\ell = 9$) [Tor13]

(d) $n, d = 3$ ($\ell = 27$) [TV21]

The same technique was also used to prove the nondefectivity of $\sigma_s(\tau(\nu_3(\mathbb{P}^n)))$ ($\ell = 24$) [AV18].

Drawback

This doesn't scale well. The $K = 3$, $\ell = 27$ case required computations involving vector spaces of dimension up to $\binom{82+3}{3} = 98{,}770$.

Identifiability

Theorem ([TV22])

Almost all $f \in \operatorname{Sym}^3 V$ with subgeneric Chow rank admit a unique Chow decomposition.

Proof.

[Oed12] $\operatorname{Ch}_d(\mathbb{P}^n)$ is not 1-tangentially weakly defective

[CM22] r-identifiable for $n \geq 103$, $r \leq \left\lfloor \frac{\binom{n+3}{3}}{3n+1} \right\rfloor - 1$

Checked remaining cases with $n \leq 102$. $\square$

Combining the two techniques

Theorem ([Tor13])

Consider the function

$$c(n,d) = \min\left\{ \left\lfloor \frac{\tilde{s}(d-m)}{g_n(m)} \right\rfloor : 0 \le m \le n-2 \right\}$$

where

$$\tilde{s}(d) = \begin{cases} \frac{1}{24}d^2 + \frac{1}{12}d & \text{if } d \equiv 0,4 \pmod 6 \\ \frac{1}{24}d^2 + \frac{1}{6}d - \frac{5}{24} & \text{if } d \equiv 1 \pmod 6 \\ \frac{1}{24}d^2 + \frac{1}{12}d - \frac{1}{3} & \text{if } d \equiv 2 \pmod 6 \\ \frac{1}{24}d^2 + \frac{1}{6}d + \frac{1}{8} & \text{if } d \equiv 3,5 \pmod 6 \end{cases}$$

and

$$g_n(m) = \begin{cases} n-3 & \text{if } m = 0 \text{ or } m = n-3 \\ n-4 & \text{if } n \ge 5 \text{ and } m = 1 \\ 1 & \text{if } m = n-2 \\ m(n-m-3) & \text{if } 2 \le m \le n-4 \end{cases}.$$

If $d \ge n$ and $s \le 2^{n-3}c(n,d)$, then $\sigma_s(\mathrm{Ch}_d(\mathbb{P}^n))$ is nondefective.

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