All secant varieties of the Chow variety are nondefective for cubics and quaternary forms

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Let $R = \mathbb{C}[x_0, \ldots, x_n]$ be a polynomial ring with the usual grading and R_d the dth graded piece of R.

Definition

If $f \in R_d$, then the **Chow rank** of f is the least s for which there exist $\ell_{i,j} \in R_1$ such that

$$f = \ell_{1,1} \cdots \ell_{1,d} + \cdots + \ell_{s,1} \cdots \ell_{s,d},$$

i.e., f may be written as the sum of s completely reducible forms.

Example

Since $x^2 + y^2 = (x + iy)(x - iy)$, its Chow rank is 1.

Consider a partition $d_1 + \cdots + d_k$ of d and $f \in R_d$. What is the least s for which there exist $\ell_{i,j} \in R_1$ such that

$$f = \ell_{1,1}^{d_1} \cdots \ell_{1,k}^{d_k} + \cdots + \ell_{s,1}^{d_1} \cdots \ell_{s,k}^{d_k}?$$

For generic f, this **Chow-Waring problem** has been solved for the following partitions of d.

- (Alexander-Hirschowitz, 1995) (d) (the *Waring problem*)
- (Abo-Vannieuwenhoven, 2018) ($d - 1, 1$)

The Chow rank case deals with the partition $(1, \ldots, 1)$.

Definition

Let X be a projective variety.
A **secant** $(s-1)$**-plane** to X is the linear subspace spanning s points of X, e.g., 2 points determine a secant line, 3 points a secant plane, etc.
The sth **secant variety** of X is the Zariski closure of the union of all $(s-1)$-planes to X, denoted $\sigma_s(X)$.

If $f, g_1, \ldots, g_s \in R_d$, $[g_i] \in X \subset \mathbb{P}R_d$, and

$$f = g_1 + \cdots + g_s,$$

then $[f] \in \sigma_s(X)$.

Definition

The **Chow variety** (aka **split variety**, **variety of completely decomposable forms**, or **variety of completely reducible forms**) is

$$\mathcal{C}_{d,n} = \{[\ell_1 \cdots \ell_d] : \ell_i \in R_1\},$$

i.e., the variety in $\mathbb{P}R_d$ corresponding to the completely reducible forms.

So the Chow rank of a generic form f is the smallest s for which

$$\sigma_s(\mathcal{C}_{d,n}) = \mathbb{P}R_d.$$

Lemma (Terracini)

Let $p_1, \ldots, p_s \in X$ *be generic. Then*

$$\dim \sigma_s(X) = \dim\langle T_{p_1}X, \ldots, T_{p_s}X\rangle.$$

We can reduce the problem of finding the dimension of a secant variety to finding the rank of a matrix!

Idea: Choose the points $p_1, \ldots, p_s$ carefully so we can use induction and semicontinuity.

The conjectured dimension when $d > 2$ is

$$\operatorname{expdim} \sigma_s(\mathcal{C}_{d,n}) = \min \left\{ s(dn+1), \binom{n+d}{d} \right\} - 1.$$

We only need to check two cases:

- *subabundant:* $s_1 = \left\lfloor \frac{1}{dn+1}\binom{n+d}{d} \right\rfloor$
- *superabundant:* $s_2 = \left\lceil \frac{1}{dn+1}\binom{n+d}{d} \right\rceil$

When we fix n or d, these are *quasipolynomial*, i.e., polynomial on congruence classes modulo some step size.

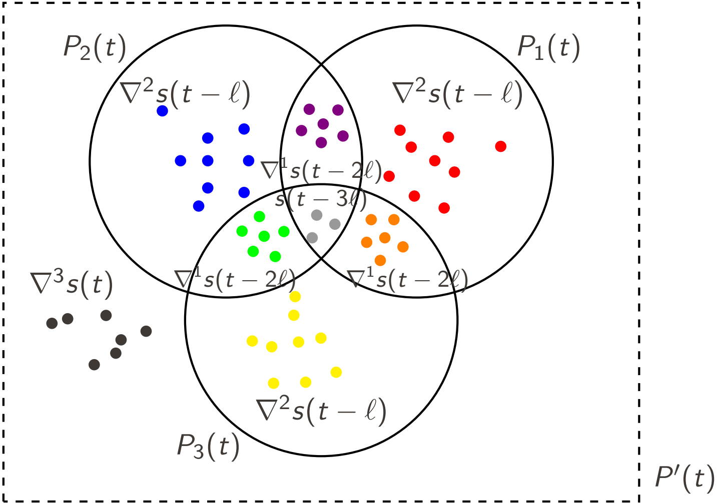
Theorem (Newton backward difference formula)

If ∇ is the backward difference operator with step size ℓ, i.e., $\nabla^0 s(t) = s(t)$ and $\nabla^i s(t) = \nabla^{i-1} s(t) - \nabla^{i-1} s(t-\ell)$, then

$$s(t) = \sum_{j=0}^{K} \binom{K}{j} \nabla^{K-j} s(t - j\ell).$$

Idea: Choose K subspaces and for each subspace in their intersection lattice, place $\nabla^{K-j} s(t-j\ell)$ of our points.

This is a *Brambilla-Ottaviani lattice*, generalizing a technique from (Brambilla-Ottaviani, 2008).


$P_2(t)$
$P_1(t)$
$\nabla^2 s(t-\ell)$
$\nabla^2 s(t-\ell)$
$\nabla^1 s(t-2\ell)$
$s(t-3\ell)$
$\nabla^3 s(t)$
$\nabla^1 s(t-2\ell)$
$\nabla^1 s(t-2\ell)$
$\nabla^2 s(t-\ell)$
$P_3(t)$
$P'(t)$

If we compute the the ranks of matrices constructed using Terracini's lemma for $t \leq K\ell + 1$ as base cases and get the expected value, then the dimension is the expected one for *all* t.

For $n = 3$ or $d = 3$, we use $K = 3$ and $\ell = 27$, requiring computations involving vector spaces of dimension up to $\binom{82+3}{3} = 98{,}770$.

```
Using random seed: 1449148218
k_0 = [2611  331 4455 1924]
l_0 = [1383 1943 1126 6474]
m_0 = [4380 1084 7722 7308]
k_1 = [1502 4383 4213 6375]
l_1 = [7679 3705 7137 5448]
m_1 = [3483 2409 7312 5909]
Constructed R(Y) in 0.01s.
Computed the rank of the 20 x 24 matrix R(Y) over F_8191
  in 0s.
Found 0 + 20 = 20 vs. 20 expected.
T(3, 2; 0, 0, 0) is TRUE (SUBABUNDANT)
Total computation took 0.011s.
```

Theorem

For all s, n, and d, $\sigma_s(\mathcal{C}_{3,d})$ and $\sigma_s(\mathcal{C}_{n,3})$ are nondefective.

Thank you!