Toy trains and polyplets

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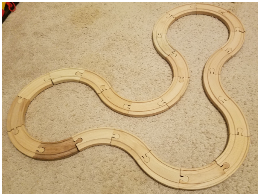
Meet Gabriel. He likes toy trains.

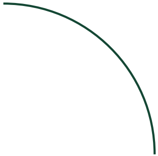
Gabriel's toy trains come with a variety of pieces of track.

Question

How many different tracks may we build with a given number of pieces?

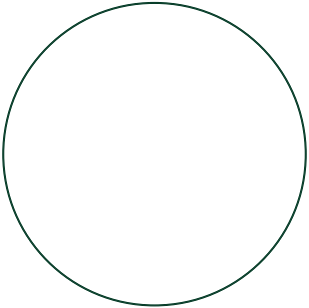


To simplify our question, let's suppose that we only have one type of track piece in the shape of a **quadrant**, or quarter circle. (Gabriel's tracks include lots of *octants*, or eighths of a circle.)

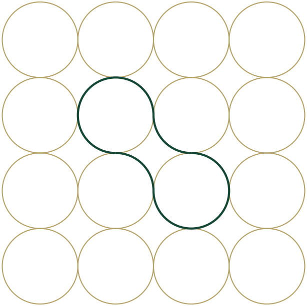


Our track is then an example of a **spline**, or a piecewise polynomial curve. Specifically, a **smooth closed regular quadrant spline**.

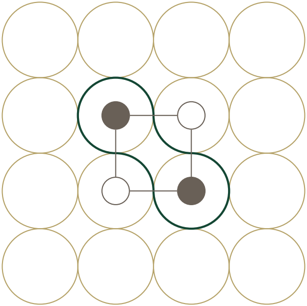
The simplest example is merely a circle.



In general, our quadrants all belong to *kissing circles* from a square circle packing of the plane.

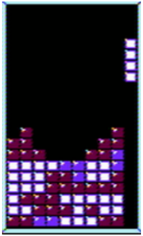


We think of each circle containing a quadrant as a vertex in a graph. And we color these vertices depending on whether they are inside or outside of our spline.



These graphs trace the boundaries of **polyplets** (or **polykings** or **pseudo-polyominos**). They are formed by gluing together squares at their edges or corners.

You're probably familiar with polyplets from, e.g., Conway's Game of Life and Tetris.

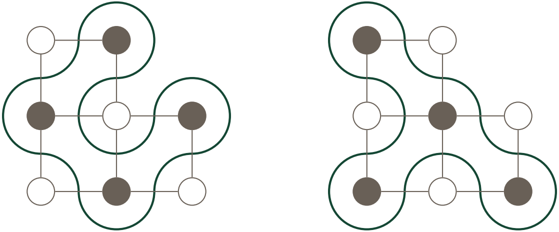


Question

Can we use polyplets to enumerate all possible smooth closed regular quadrant splines?

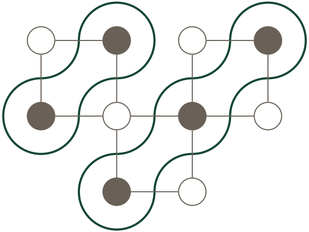
We want to enumerate them up to *similarity*, e.g., the splines below are considered the same.

Some polyplets correspond to more than one spline.



L-triomino or *pre-block*

And other polyplets result in disconnected splines.

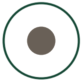


To get around these issues, we impose some conditions.

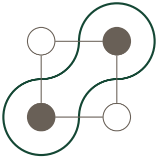
- We look at both possible vertex 2-colorings of the boundary of each polyplet. These may be isomorphic.
- We only consider polyplets for which a vertex 2-coloring colors all cut-vertices the same. These cut-vertices must correspond to a circle inside the spline.
- As we only care about the boundary, we only consider polyplets without holes.

We can now begin enumerating all smooth closed regular quadrant splines.

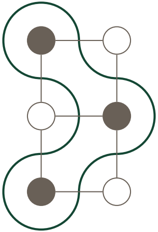
0-plet:

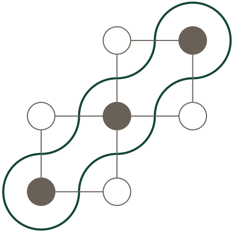


1-plet:

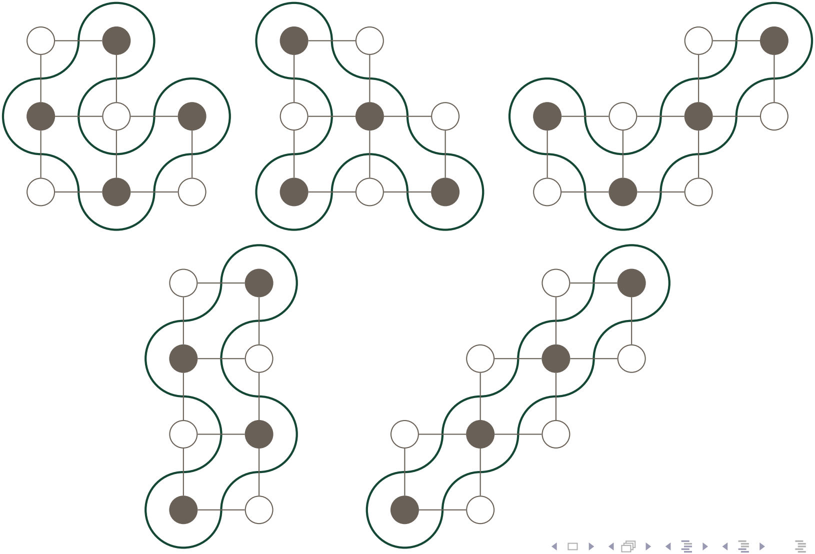


2-plets:





3-plets:

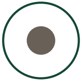


Theorem

If a smooth closed regular quadrant spline corresponds to an n-plet, then it consists of $4k$ quadrants for some positive integer $k \leq n + 1$.

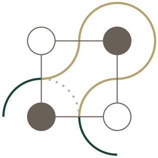
Proof. We use induction on n.

For the base case, the only spline corresponding to a 0-plet is a circle, which consists of 4 quadrants.

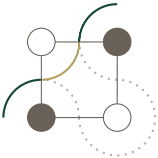


Now suppose we add a square to an n-plet to form an $(n+1)$-plet. There are essentially two cases.

Case 1. We remove 1 quadrant and add 5, and so we increase the total number of quadrants by 4.



Case 2. We remove 5 quadrants and add 1, and so we decrease the total number of quadrants by 4.



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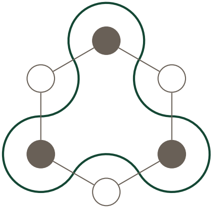
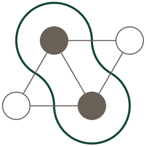
Why stop with quadrants and polyplets?

We can build **trient** splines using **polyhexes**.



Or **sextant** splines using **polyiamonds**.



Beyond these, it gets ugly – we’ve run out of regular uniform circle packings.

Thank you!