Triangular and hexagonal Tangles

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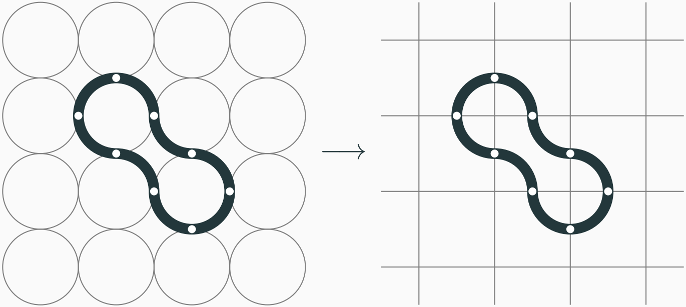
Tangles

A **Tangle** (named after the popular fidget toy) is a smooth closed plane curve made by gluing together arcs of circles with a common radius.

The existing literature is devoted to **square Tangles**, in which the arcs are all quarter circles like the toys.

Observation

The arcs in a square Tangle all belong to circles in a *square circle packing*, which corresponds to the *square tiling* of the plane.



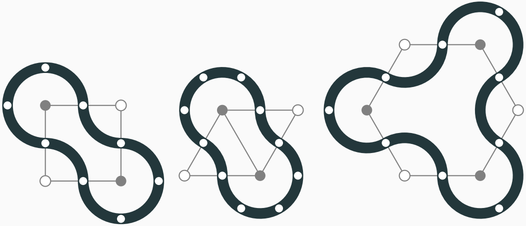
Idea

There are two other *regular* tilings of the plane: by equilateral triangles and regular hexagons. Build Tangles using them!

We call a Tangle constructed from one of the three regular tilings a **regular Tangle**.

Dual polyforms

Every regular Tangle has a **dual polyform**, which consists of a connected union of polygons from the underlying tiling together with a 2-coloring of its boundary (black = inside, white = outside).



- square Tangle ($\frac{1}{4}$-circle links) $\longrightarrow$ dual pseudo-polyomino
- triangular Tangle ($\frac{1}{6}$-circle links) $\longrightarrow$ dual pseudo-polyiamond
- hexagonal Tangle ($\frac{1}{3}$-circle links) $\longrightarrow$ dual polyhex

Size

The **size** of a regular Tangle is the number of polygons in its dual polyform.

On the previous slide, the Tangles have sizes 1, 2, and 1, resp.

Length

The **length** of a Tangle is the number of arcs (or **links**) that comprise it.

It is known that the length of a square Tangle is always a multiple of 4 [Fle00].

Question

What can we say about the lengths of triangular and hexagonal Tangles?

Area

The area enclosed by a square Tangle with size m and radius r is $(4m + \pi)r^2$ [Bro08, Tor20].

Question

What can we say about the areas enclosed by triangular and hexagonal Tangles?

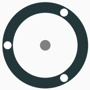
Hexagonal Tangle length

Theorem ([Bow20])

If a hexagonal Tangle has length n, then $n \equiv 3 \pmod 6$.

The proof uses induction on the size m.

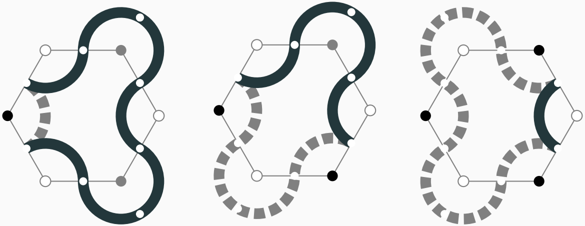
Base case. If $m = 0$, then we have a circle, which has length 3.



Inductive step

Add one hexagon to an existing dual polyhex to obtain a new dual polyhex. There are three cases:

1. *(shear insertion)* $+7 - 1 = +6$ links
2. *(reflection)* $+4 - 4 = +0$ links
3. *(shear reduction)* $1 - 7 = -6$ links



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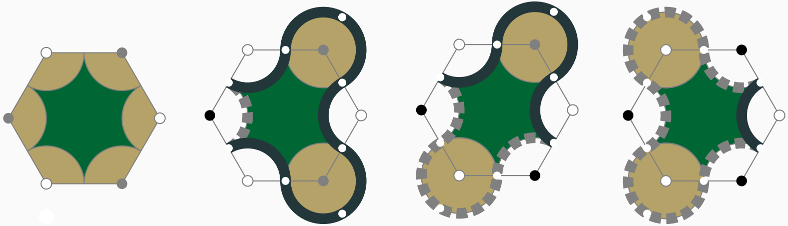
Hexagonal Tangle area

Theorem

If a hexagonal Tangle has size m and radius r, then it encloses an area of $(6\sqrt{3}m + \pi)r^2$.

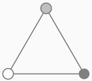
We again use induction on m. When $m = 0$, we have a circle with area πr^2. Each operation adds exactly the area of one hexagon ($6\sqrt{3}r^2$).



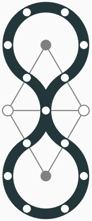
Triangular Tangles

Triangular Tangles are trickier:

- Boundaries of pseudo-polyiamonds are not necessarily 2-colorable.

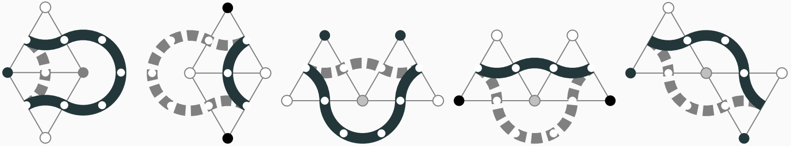


- Some pseudo-polyiamonds correspond to singular curves.



Triangular Tangle operations

There are five basic operations on triangular Tangles that result from adding a pseudo-diamond to the dual pseudo-polyiamond. A triangular Tangle that can be obtained from a circle using only these operations is **constructible**.



Conjecture

Every triangular Tangle is constructible.

Triangular Tangle length

Theorem ([Pru20])

If a constructible triangular Tangle has length n, then n is even.

The proof uses induction on the size m.

Base case. If $m = 0$, then we have a circle, which has length 6.

Inductive step. Each of the five operations result in the addition or removal of an even number of links.

1. *(diamond shear insertion)* $+6 - 2 = +4$ links
2. *(diamond shear reduction)* $+2 - 6 = -4$ links
3. *(butterfly shear insertion)* $+5 - 3 = +2$ links
4. *(butterfly shear reduction)* $+3 - 5 = -2$ links
5. *(reflection)* $+4 - 4 = +0$ links

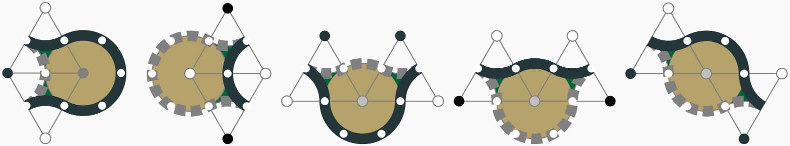
□

Triangular Tangle area

Theorem

If a constructible triangular Tangle has size m and radius r, then it encloses an area of $(\sqrt{3}m + \pi)r^2$.

We again use induction on m. When $m = 0$, we have a circle with area πr^2. Each operation (which increases the size by 2) adds exactly the area of two triangles ($\sqrt{3}r^2$ each).



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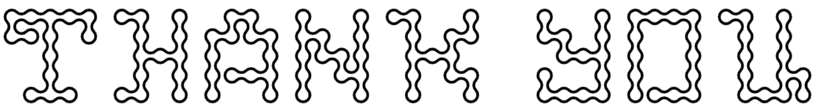

Tangle font: `http://erikdemaine.org/fonts/tangle/`

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