Nondefective secant varieties of varieties of completely decomposable forms

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Waring's Problem

Question

For a given $d \in \mathbb{N}$, what is the smallest $s \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, there exist $n_1, \ldots, n_s \in \mathbb{Z}$ such that

$$n = n_1^d + \cdots + n_s^d?$$

Example (Lagrange's Four Square Theorem – 1770)

If $d = 2$, then $s = 4$.

Waring's Problem for polynomials

Let $\Bbbk$ be an algebraically closed field of characteristic 0 and let $R = \Bbbk[x_0, \dots, x_n]$. Let R_d denote the homogeneous polynomials of degree d in R.

Question

For a given $d \in \mathbb{N}$, what is the smallest $s \in \mathbb{N}$ such that, for any generic $f \in R_d$, there exist $\ell_1, \dots, \ell_s \in R_1$ such that

$$f = \ell_1^d + \cdots + \ell_s^d?$$

This question was answered by Alexander and Hirschowitz in a series of papers from 1992–5.

Another Waring's Problem for polynomials

We say that $f \in R_d$ is *completely decomposable* if $f = \ell_1 \cdots \ell_d$, where $\ell_i \in R_1$ for all i.
We can therefore generalize the Waring's Problem for polynomials in the following way:

Question

For a given $d \in \mathbb{N}$, what is the smallest $s \in \mathbb{N}$ such that, for any generic $f \in R_d$, there exist completely decomposable $f_1, \ldots, f_s \in R_d$ such that

$$f = f_1 + \cdots + f_s?$$

Varieties of completely decomposable forms

Definition

A *variety of completely decomposable forms* or *Chow variety of zero cycles* is a variety of the form

$$\mathrm{Split}_d(\mathbb{P}^n) = \{[f] \in \mathbb{P}R_d : f \text{ is completely decomposable}\}$$

Note that $\mathrm{Split}_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{d}-1}$.

Secant varieties

Definition

Let $X \subset \mathbb{P}V$ be a variety. The sth *secant variety* of X is

$$\sigma_s(X) = \overline{\bigcup \{\langle p_1, \ldots, p_s \rangle : p_1, \ldots, p_s \in X\}}$$

Here, $\bar{\cdot}$ denotes Zariski closure and $\langle \cdot \rangle$ denotes linear span.
Note that some other authors use $\sigma_{s-1}(X)$ instead.

Waring problem, revisited

We can now reframe our original question:

Question

For a given $d \in \mathbb{N}$, what is the smallest $s \in \mathbb{N}$ such that $\dim \sigma_s(\mathrm{Split}_d(\mathbb{P}^n)) = \binom{n+d}{d} - 1$?

Big question

Question

What is $\dim \sigma_s(\mathrm{Split}_d(\mathbb{P}^n))$?.

Note that

$$\begin{aligned}\dim \sigma_s(\mathrm{Split}_d(\mathbb{P}^n)) &\leq \operatorname{expdim} \sigma_s(\mathrm{Split}_d(\mathbb{P}^n)) \\ &= \min\left\{s(dn+1), \binom{n+d}{d}\right\} - 1\end{aligned}$$

If $\dim \sigma_s(\mathrm{Split}_d(\mathbb{P}^n)) = \operatorname{expdim} \sigma_s(\mathrm{Split}_d(\mathbb{P}^n))$, the variety is *nondefective* and we are done. Otherwise, it is *defective*. We may rephrase our question:

Question

When is $\sigma_s(\mathrm{Split}_d(\mathbb{P}^n))$ defective?

Calculating dimension using Terracini's lemma

Lemma (Terracini – 1911)

Choose s generic points $p_1, \ldots, p_s$ on a projective variety X. Then, for a generic point $q \in \langle p_1, \ldots, p_s \rangle$,

$$T_q \sigma_s(X) = \langle T_{p_1} X, \ldots, T_{p_s} X \rangle$$

Note that, for a generic $f = \ell_1 \cdots \ell_d$, $\ell_i \in R_1$, we have

$$\widehat{T}_{[f]} \operatorname{Split}_d(\mathbb{P}^n) = \sum_{j=1}^{d} \ell_1 \cdots \ell_{j-1} \ell_{j+1} \cdots \ell_d R_1$$

So calculating the dimension of our variety is equivalent to calculating the dimension of a vector space.

Terracini's lemma example

Suppose we want to calculate the dimension of $\sigma_2(\text{Split}_3(\mathbb{P}^5))$. We choose the tangent spaces to the points $[x_0x_1x_2]$ and $[x_3x_4x_5]$. Then we just need to find the dimension of the vector space

$$x_0x_1R_1 + x_0x_2R_1 + x_1x_2R_1 + x_3x_4R_1 + x_3x_5R_1 + x_4x_5R_1,$$

which can be calculated by finding the rank of a 56×32 matrix.

Conjecture

Conjecture (Arrondo, Bernardi – 2011)

The secant variety $\sigma_s(\text{Split}_d(\mathbb{P}^n))$ is defective if and only if $d = 2$ and $2 \leq s \leq \frac{n}{2}$.

The $\Leftarrow$ direction can be proven using Terracini's lemma.
The $\Rightarrow$ direction is far more difficult.

Known results

Theorem

If any of the following are true, then $\sigma_s(\mathrm{Split}_d(\mathbb{P}^n))$ is nondefective.

- $n = 1$ *or* $d = 1$ *(trivial cases)*
- $d = 2$ *and* $s > \frac{n}{2}$ *(application of Terracini's lemma)*
- $d \geq 3$ *and* $n \geq 3(s-1)$ *(Arrondo-Bernardi – 2011)*
- $n = 2$ *(Shin – 2011, Abo – 2012)*
- $n = 3$, $s \leq s_1(d)$ *or* $s \geq s_2(d)$ *for some functions* s_1, s_2 *(Abo – 2012)*
- $d = 3$, $s \leq s_1(n)$ *or* $s \geq s_2(n)$ *for these same functions (Abo – 2012)*

Can we fill in the gaps?

Splitting induction

We use two methods of induction. In the first, we use the fact that

$$R_d = x_0 R_{d-1} \oplus S_d U$$

where $U = \text{span}\{x_1, \ldots, x_n\}$
In particular, we look at specialized points in $\text{Split}_d(\mathbb{P}^n)$, and split the vector space from Terracini's lemma into the direct sum of smaller vector spaces.

Splitting induction example

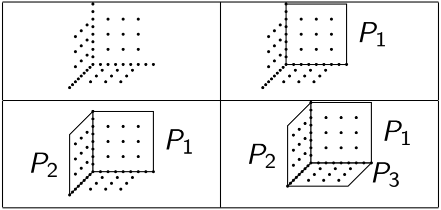
For example, consider our $\sigma_2(\mathrm{Split}_3\,\mathbb{P}^5)$ example from above.

$$\begin{aligned}&x_0x_1R_1 + x_0x_2R_1 + x_1x_2R_1 + x_3x_4R_1 + x_3x_5R_1 + x_4x_5R_1\\ &\quad = (x_1x_2U + x_3x_4U + x_3x_5U + x_4x_5U)\\ &\qquad \oplus x_0(x_1U + x_2U + \mathrm{span}\{x_3x_4, x_3x_5, x_4x_5\})\\ &\qquad \oplus x_0^2\,\mathrm{span}\{x_1, x_2\}\end{aligned}$$

We need only find the dimensions of each direct summand.

Restriction induction

Recall that the vector space we use to calculate the dimension of the secant varieties is a sum of tangent spaces at specific points. We restrict these points to smaller varieties of completely decomposable forms (either less variables or smaller degree), and use the dimensions of these smaller cases to find the dimension of the larger case.



$n = 3$

Theorem (Torrance – 2013)

Let

$$s_1(d) = \begin{cases} \frac{1}{18}d^2 + \frac{5}{18}d & \text{if } d \equiv 0, 4 \pmod 9 \\ \frac{1}{18}d^2 + \frac{5}{18}d + \frac{2}{9} & \text{if } d \equiv 2, 5, 8 \pmod 9 \\ \frac{1}{18}d^2 + \frac{5}{18}d + \frac{2}{3} & \text{if } d \equiv 1, 3 \pmod 9 \\ \frac{1}{18}d^2 + \frac{5}{18}d + \frac{1}{3} & \text{if } d \equiv 6, 7 \pmod 9 \end{cases}$$

$$s_2(d) = \begin{cases} \frac{1}{18}d^2 + \frac{1}{3}d + 1 & \text{if } d \equiv 0 \pmod 6 \\ \frac{1}{18}d^2 + \frac{1}{3}d + \frac{1}{2} & \text{if } d \equiv 3 \pmod 6 \\ \frac{1}{18}d^2 + \frac{7}{18}d + \frac{5}{9} & \text{if } d \equiv 1, 4, 7 \pmod 9 \\ \frac{1}{18}d^2 + \frac{7}{18}d + 1 & \text{if } d \equiv 2 \pmod 9 \\ \frac{1}{18}d^2 + \frac{7}{18}d + \frac{2}{3} & \text{if } d \equiv 5 \pmod 9 \\ \frac{1}{18}d^2 + \frac{7}{18}d + \frac{1}{3} & \text{if } d \equiv 8 \pmod 9 \end{cases}$$

If $s \leq s_1(d)$ or $s \geq s_2(d)$, then $\sigma_s(\mathrm{Split}_d(\mathbb{P}^3))$ is nondefective for all $d \in \mathbb{N}$

$n = 3$ Proof

Proof. The s_1 and s_2 functions are as close as possible to $\frac{\binom{n+d}{d}}{dn+1}$, the point at which a secant variety switches from *subabundant* (not expected to fill the ambient space) to *superabundant* (expected to fill the ambient space), while still allowing restriction induction to be used with a relatively small step size. The base cases were confirmed in Macaulay2. □

$n \geq 4$

Theorem (Torrance – 2013)

Consider the function

$$c(n,d) = \min\left\{ \left\lfloor \frac{\tilde{s}(d-m)}{g_n(m)} \right\rfloor : 0 \leq m \leq n-2 \right\}$$

where

$$\tilde{s}(d) = \begin{cases} \frac{1}{24}d^2 + \frac{1}{12}d & \text{if } d \equiv 0,4 \pmod 6 \\ \frac{1}{24}d^2 + \frac{1}{6}d - \frac{5}{24} & \text{if } d \equiv 1 \pmod 6 \\ \frac{1}{24}d^2 + \frac{1}{12}d - \frac{1}{3} & \text{if } d \equiv 2 \pmod 6 \\ \frac{1}{24}d^2 + \frac{1}{6}d + \frac{1}{8} & \text{if } d \equiv 3,5 \pmod 6 \end{cases}$$

and

$$g_n(m) = \begin{cases} n-3 & \text{if } m = 0 \text{ or } m = n-3 \\ n-4 & \text{if } n \geq 5 \text{ and } m = 1 \\ 1 & \text{if } m = n-2 \\ m(n-m-3) & \text{if } 2 \leq m \leq n-4 \end{cases}.$$

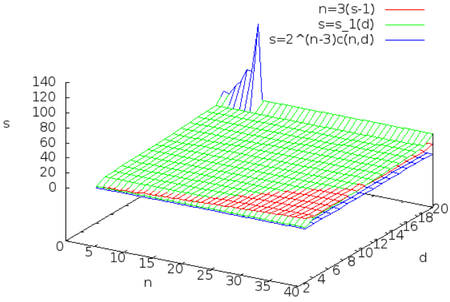
If $d \geq n$ and $s \leq \max\{s_1(d), 2^{n-3}c(n,d)\}$, then $\sigma_s(\mathrm{Split}_d(\mathbb{P}^n))$ is nondefective.

$n \geq 4$ Proof

Proof. Splitting induction can be used to extend the subabundant case from the previous theorem to $n \geq 4$, giving the $s \leq s_1(d)$ bound.
Also, if $d \gg n$, we may use splitting induction to split a vector space into a number of smaller vector spaces with $n = 3$. We can then use restriction induction to find the dimensions of these vector spaces, which in turn tells us the dimension of the original vector space. This gives us the $s \leq 2^{n-3}c(n, d)$ bound. □

Comparison of results

The following graph shows these two bounds as compared with the known bound of Arrondo and Bernardi.



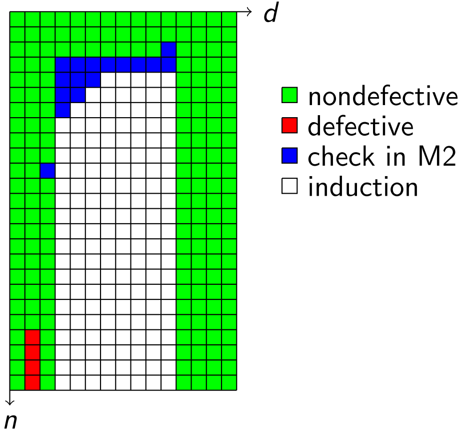
Small s

Theorem (Torrance – 2013)

If $s \leq 30$, then $\sigma_s(\text{Split}_d(\mathbb{P}^n))$ is nondefective for all $n, d \in \mathbb{N}$ unless $d = 2$ and $n \geq 2s$.

Small s Proof

Proof. After applying the above results, there exist only finitely many cases to check for any fixed s. Each case was checked in Macaulay2, for as large an s as possible. □



Thank you!

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