An exponential bound for nondefective secant varieties of Chow varieties

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Chow varieties

Let V be a finite dimensional $\mathbb{C}$-vector space. For a fixed d, consider the map

$$\mathrm{Ch}_d : \mathbb{P}(V)^{\times d} \to \mathbb{P}(\mathrm{Sym}^d V)$$

$$([\ell_1], \ldots, [\ell_d]) \to [\ell_1 \cdots \ell_d]$$

The image of this map, which we denote $\mathrm{Ch}_d(\mathbb{P}^n)$ if $\dim V = n + 1$, is a *Chow variety* (of 0-cycles), *split variety*, or *variety of complete decomposable (or reducible) forms*.

Secant varieties

Given a projective variety X and nonnegative $r \in \mathbb{Z}$, the sth *secant variety* of X is

$$\sigma_s(X) = \overline{\{\langle p_1, \ldots, p_s \rangle : p_1, \ldots, p_s \in X\}}.$$

For the smallest s such that $\sigma_s(\mathrm{Ch}_d(\mathbb{P}^n)) = \mathbb{P}(\mathrm{Sym}^d V)$,

$$f = \sum_{i=1}^{s} \ell_{i,1} \cdots \ell_{i,d}$$

for all generic $(n+1)$-ary d-ics f.

Computational complexity: This decomposition gives us an efficient way of evaluating f.

Conjecture

For all n, d, s,

$$\dim \sigma_s(\mathrm{Ch}_d(\mathbb{P}^n)) = \min\left\{ s(dn+1), \binom{n+d}{d} \right\} - 1$$

except for the cases $d = 2$, $n \geq 4$, $2 \leq s \leq \frac{n}{2}$.

Terracini's lemma

Lemma

For any n, d, s*,*

$$\dim \sigma_s(\mathrm{Ch}_d(\mathbb{P}^n)) = \dim \sum_{i=1}^{s} \sum_{j=1}^{d} \ell_{i,1} \cdots \ell_{i,j-1} \ell_{i,j+1} \cdots \ell_{i,d} V - 1$$

for generic $\ell_{i,j} \in V$*.*

Abo-Ottaviani-Peterson induction

Theorem (– (2017))

Except for the known defective cases, $\sigma_s(\mathrm{Ch}_d(\mathbb{P}^n))$ has the expected dimension if $s \leq 35$.

Brambilla-Ottaviani induction

Theorem (–, Vannieuwenhoven (2021))

Except for the known defective cases, $\sigma_s(\mathrm{Ch}_d(\mathbb{P}^n))$ has the expected dimension if $n = 3$ or $d = 3$.

Combining the two techniques

Theorem (–)

Consider the function

$$c(n,d) = \min\left\{ \left\lfloor \frac{s(d-m)}{g_n(m)} \right\rfloor : 0 \le m \le n-2 \right\}$$

where

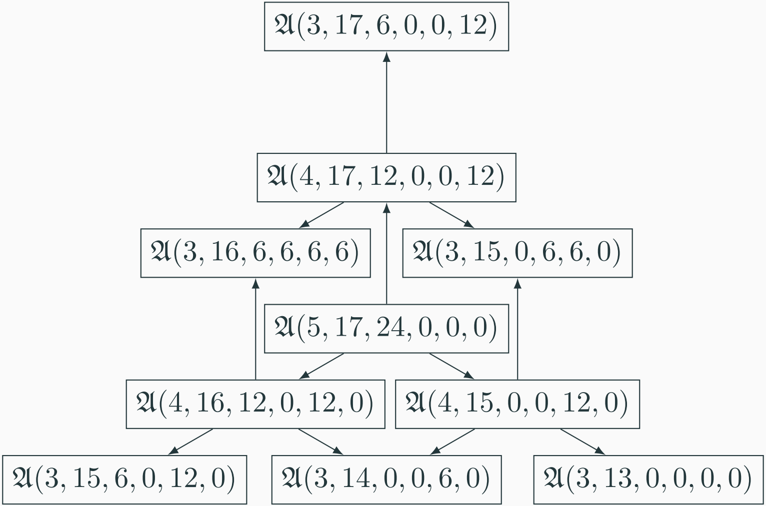
$$s(d) = \begin{cases} \frac{1}{24}d^2 + \frac{1}{12}d & \text{if } d \equiv 0,4 \pmod 6 \\ \frac{1}{24}d^2 + \frac{1}{6}d - \frac{5}{24} & \text{if } d \equiv 1 \pmod 6 \\ \frac{1}{24}d^2 + \frac{1}{12}d - \frac{1}{3} & \text{if } d \equiv 2 \pmod 6 \\ \frac{1}{24}d^2 + \frac{1}{6}d + \frac{1}{8} & \text{if } d \equiv 3,5 \pmod 6 \end{cases}$$

and

$$g_n(m) = \begin{cases} n-3 & \text{if } m = 0 \text{ or } m = n-3 \\ n-4 & \text{if } n \ge 5 \text{ and } m = 1 \\ 1 & \text{if } m = n-2 \\ m(n-m-3) & \text{if } 2 \le m \le n-4 \end{cases}.$$

If $d \ge n \ge 4$ and $s \le 2^{n-3}c(n,d)$, then $\sigma_s(\mathrm{Ch}_d(\mathbb{P}^n))$ is nondefective.

Proof idea


$\mathfrak{A}(3,17,6,0,0,12)$
$\mathfrak{A}(4,17,12,0,0,12)$
$\mathfrak{A}(3,16,6,6,6,6)$
$\mathfrak{A}(3,15,0,6,6,0)$
$\mathfrak{A}(5,17,24,0,0,0)$
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$\mathfrak{A}(3,13,0,0,0,0)$

Thank you!

https://webwork.piedmont.edu/~dtorrance/research