The Gauss-Bonnet theorem and triangular Tangles

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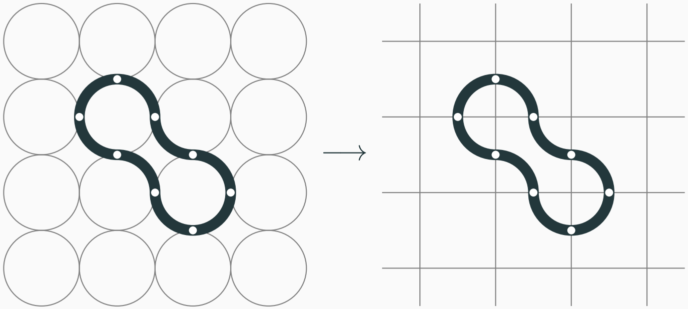
Tangles

A **Tangle** (named after the popular fidget toy) is a smooth (C^1, piecewise C^∞) closed plane curve made by gluing together arcs of circles with a common radius (which we call *links* of the Tangle).

The Tangle toy is made of quarter circle links, but we can use other proportions as well!

Observation

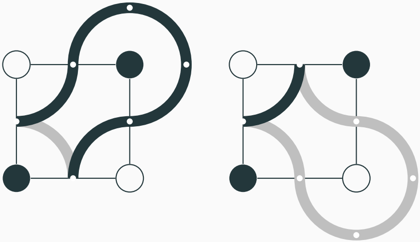
The links of a Tangle all belong to circles in a *circle packing* of the plane, which corresponds to a *tiling* of the plane.



There are 11 tilings of the plane by convex regular polygons (square, hexagonal, triangular, and 8 more *semiregular* tilings). All give rise to families of Tangles!

Fundamental operations

For each family of Tangles, we identify a set of *fundamental operations*, each of which involves the addition of polygon(s) from the underlying tiling.



Shear insertion and shear reduction, the two fundamental operations on square Tangles.

Constructible Tangles

A Tangle is *constructible* if it may be constructed from a circle using a sequence of fundamental operations.

constructible Tangles $\implies$ use induction for interesting results!

Theorem (Bowen, Pruitt, T.)

If a constructible Tangle has radius r, length ℓ, and size m, and encloses area A, then

- *(square)* $\ell \equiv 0 \pmod 4$, $A = (4m + \pi)r^2$
- *(hexagonal)* $\ell \equiv 3 \pmod 6$, $A = (6\sqrt{3}m + \pi)r^2$
- *(triangular)* $\ell \equiv 0 \pmod 2$, $A = (\sqrt{3}m + \pi)r^2$

Is every Tangle constructible?

- square – yes!
- hexagonal – yes!
- triangular – maybe??

Gauss-Bonnet theorem

Theorem (Gauss-Bonnet)

If M is a compact 2-manifold with Gaussian curvature K and Euler characteristic $\chi(M)$ and ∂M is its boundary with geodesic curvature k_g, then

$$\iint_M K\,dA + \oint_{\partial M} k_g\,ds = 2\pi\chi(S).$$

Very special case: the angles in a triangle sum to 180°.

Theorem (T.)

Suppose $0 < p_i < 1$ *for all* $i = 1, \ldots, r$. *If all the links of a Tangle are* p_i*-circles for some* i, *and if for each* i *there are* d_i *more convex* p_i*-links than concave* p_i*-links, then*

$$p_1 d_1 + \cdots + p_r d_r = 1.$$

Proof

Let M be the interior of the Tangle. Since $M \subset \mathbb{R}^2$, $K = 0$ and $\chi(M) = 1$.

Each link has geodesic curvature $\pm\frac{1}{r}$ ($+$: convex, $-$: concave). So Gauss-Bonnet gives us

$$\iint_M 0\, dA + \sum_{i=1}^{r} d_i \oint_{p_i\text{-link}} \frac{1}{r}\, ds = 2\pi \cdot 1$$

$$\sum_{i=1}^{r} d_i \cdot \frac{1}{r} \cdot 2\pi r p_i = 2\pi$$

$$\sum_{i=1} p_i d_i = 1 \qquad \square$$

Gauss-Bonnet theorem for regular Tangles ($r = 1$)

tiling	p_1	d_1
square [Fleron]	$\frac{1}{4}$	4
hexagonal	$\frac{1}{3}$	3
triangular	$\frac{1}{6}$	6

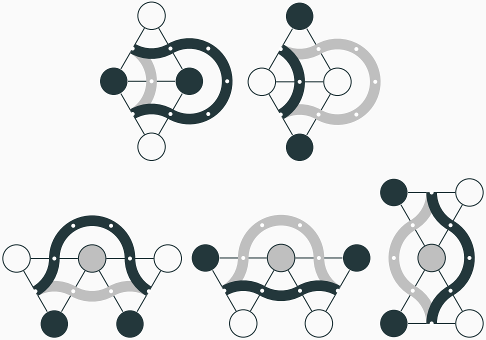
Gauss-Bonnet theorem for semiregular Tangles ($r = 2$)

tiling	(p_1, p_2)	(d_1, d_2)
snub square, elongated triangular	$\left(\frac{1}{6}, \frac{1}{4}\right)$	$(0,4)$, $(3,2)$, $(6,0)$
trihexagonal, snub hexagonal	$\left(\frac{1}{6}, \frac{1}{3}\right)$	$(0,3)$, $(2,2)$, $(4,1)$, $(6,0)$
truncated hexagonal	$\left(\frac{1}{6}, \frac{5}{12}\right)$	$(1,2)$, $(6,0)$
truncated square	$\left(\frac{1}{4}, \frac{3}{8}\right)$	$(1,2)$, $(4,0)$

Gauss-Bonnet theorem for semiregular Tangles ($r = 3$)

tiling	(p_1, p_2, p_3)	(d_1, d_2, d_3)
rhombitrihexagonal	$\left(\frac{1}{6}, \frac{1}{4}, \frac{1}{3}\right)$	$(0,0,3),\ (0,4,0),$ $(1,2,1),\ (2,0,2),$ $(3,2,0),\ (4,0,1),$ $(6,0,0)$
truncated trihexagonal	$\left(\frac{1}{4}, \frac{1}{3}, \frac{5}{12}\right)$	$(0,3,0),\ (1,1,1),$ $(4,0,0)$

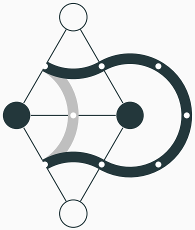
Triangular Tangle fundamental operations



4-bulb shear insertion (top left), 4-bulb shear reduction (top right), 3-bulb shear insertion (bottom left), 3-bulb shear reduction (bottom center), reflection (bottom right)

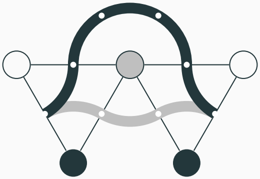
Triangular Tangles are constructible (step 1)

If there exist non-constructible triangular Tangles, then there exists one of minimal length.



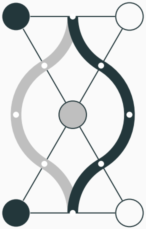
It can't have a 4-bulb, or otherwise we can construct it from a constructible Tangle via 4-bulb shear insertion.

Triangular Tangles are constructible (step 2)



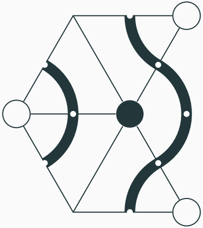
It can't have a 3-bulb, or otherwise we can construct it from a construcible Tangle via 3-bulb shear insertion.

Triangular Tangles are constructible (step 3)



It can't contain a 2-bulb that could be obtained via reflection from a constructible Tangle

Triangular Tangles are constructible (step 4)



So our Tangle must contain only 1-bulbs or 2-bulbs like the one shown above. It follows that we must have at least as many concave links as convex links, contradicting Gauss-Bonnet. □

THANK YOU

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https://webwork.piedmont.edu/~dtorrance/research