Generic forms of low Chow rank

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Let $\mathbb{k}$ be an algebraically closed field of characteristic 0, $R = \mathbb{k}[x_0, \ldots, x_n]$ a polynomial ring with the usual grading, and R_d the dth graded piece of R.

Definition

If $f \in R_d$, then the **Chow rank** of f is the least s for which there exist $\ell_{i,j} \in R_1$ such that

$$f = \ell_{1,1} \cdots \ell_{1,d} + \cdots + \ell_{s,1} \cdots \ell_{s,d},$$

i.e., f may be written as the sum of s completely reducible forms.

Example

Since $x^2 - y^2 = (x+y)(x-y)$, its Chow rank is 1.

The Chow rank tells us something about the computational complexity of evaluating a form.

Example

Suppose $f(x, y) = x^2 - y^2$. Then

$$f(2, 1) = 2 \times 2 - 1 \times 1 = 4 - 1 = 3,$$

which requires 2 multiplications. But also

$$f(2, 1) = (2 + 1) \times (2 - 1) = 3 \times 1 = 3,$$

only requiring 1 multiplication.

Definition

Let X be a projective variety.
A **secant** $(s-1)$**-plane** to X is the linear subspace spanning s points of X, e.g., 2 points determine a secant line, 3 points a secant plane, etc.
The sth **secant variety** of X is the Zariski closure of the union of all $(s-1)$-planes to X, denoted $\sigma_s(X)$.

If $f, g_1, \dots, g_s \in R_d$, $[g_i] \in X \subset \mathbb{P}R_d$, and

$$f = g_1 + \cdots + g_s,$$

then $[f] \in \sigma_s(X)$.

Definition

The **Chow variety** (aka **split variety**, **variety of completely decomposable forms**, or **variety of completely reducible forms**) is

$$\mathrm{Split}_d(\mathbb{P}^n) = \{[\ell_1 \cdots \ell_d] : \ell_i \in R_1\},$$

i.e., the variety in $\mathbb{P}R_d$ corresponding to the completely reducible forms.

So the Chow rank of a generic form f is the smallest s for which

$$\sigma_s(\mathrm{Split}_d(\mathbb{P}^n)) = \mathbb{P}R_d.$$

Lemma (Terracini)

Let $p_1, \ldots, p_s \in X$ be generic. Then

$$\dim \sigma_s(X) = \dim \langle T_{p_1} X, \ldots, T_{p_s} X \rangle.$$

We can reduce the problem of finding the dimension of a secant variety to finding the rank of a matrix!

Suppose A is the matrix whose rank determines the dimension of $\sigma_s(\mathrm{Split}_d(\mathbb{P}^n))$. By careful choice of our points $p_1, \ldots, p_s$, we can find matrices B, C, and D corresponding to spaces of forms with n variables and degrees d, $d-1$, and $d-2$, respectively, such that

$$A = \begin{pmatrix} B & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & D \end{pmatrix}.$$

Then

$$\operatorname{rank} A = \operatorname{rank} B + \operatorname{rank} C + \operatorname{rank} D.$$

Using induction, we obtain the following result.

Theorem (T.)

If

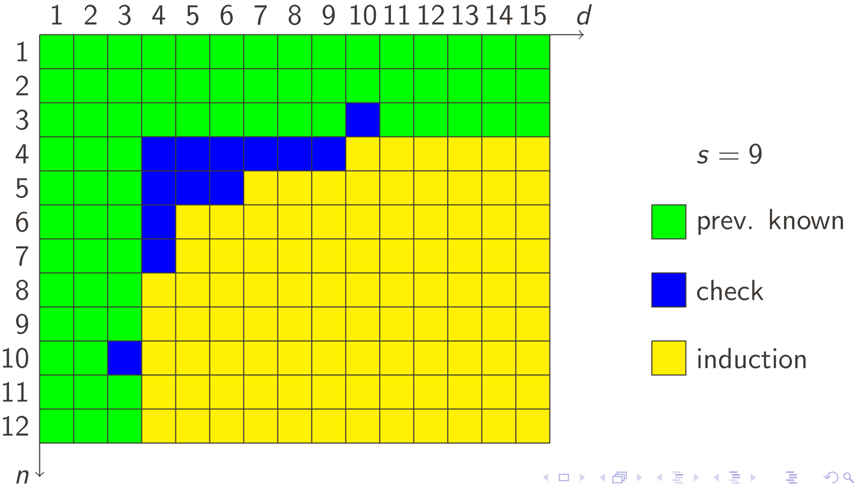
$$\dim \sigma_s(\mathrm{Split}_d(\mathbb{P}^{n_0})) = s(dn_0 + 1) - 1,$$

then

$$\dim \sigma_s(\mathrm{Split}_d(\mathbb{P}^{n})) = s(dn + 1) - 1$$

for all $n \geq n_0$.

For fixed s, this reduces finding $\dim \sigma_s(\mathrm{Split}_d(\mathbb{P}^n))$ for all n, d to checking finitely many base cases.



Using Macaulay2 to check as many of these base cases as possible, we obtain the following result.

Theorem (T.)

If $s \leq 35$, then

$$\dim \sigma_s(\mathrm{Split}_d(\mathbb{P}^n)) = \min \left\{ s(dn+1), \binom{n+d}{d} \right\} - 1,$$

except for some previously known special cases when $d = 2$.

Thank you!