Biconal subspace arrangements

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Definition

Suppose $R = \mathbb{k}[x_0, \ldots, x_n]$ for some field $\mathbb{k}$ and let R_1 be the vector space of *linear forms*, i.e., degree 1 homogeneous elements of R. If $\ell_1, \ldots, \ell_c \in R_1$ are linearly independent, then the set

$$V(\ell_1, \ldots, \ell_c) = \{P \in \mathbb{P}^n : \ell_i(P) = 0 \text{ for all } i\}$$

is a *linear subspace* of $\mathbb{P}^n$ of *codimension* c.

Think lines in $\mathbb{P}^3$ or planes in $\mathbb{P}^4$.

Consider two distinct linear subspaces of codimension 2. They either intersect in codimension 3 or codimension 4. (Linear algebra!)

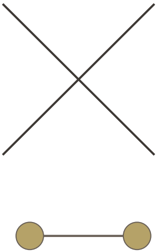
- $V(x, y) \cap V(y, z) = V(x, y, z)$
- $V(x, y) \cap V(z, w) = V(x, y, z, w)$

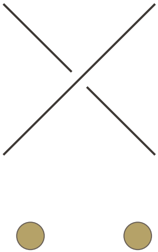
In $\mathbb{P}^3$, lines intersects in points or not at all.
In $\mathbb{P}^4$, planes intersect in lines or points.

Definition

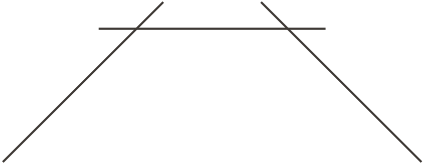
Let $\mathcal{A}$ be an arrangement of linear subspaces of codimension 2. The *incidence graph* of $\mathcal{A}$ is the graph $\Gamma(\mathcal{A})$ with

- vertex set $\mathcal{A}$
- edge set $\{\{V_1, V_2\} : \operatorname{codim} V_1 \cap V_2 = 3\}$



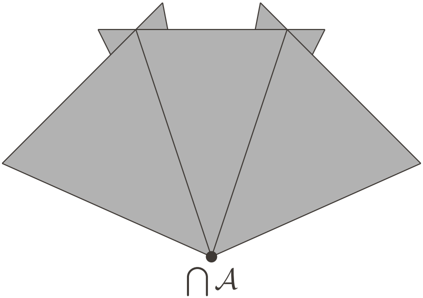


Consider a line arrangement $\mathcal{A} = \{V_1, V_2, V_3\}$ in $\mathbb{P}^3$ with $\Gamma(\mathcal{A}) = P_3$.



Note that $\bigcap \mathcal{A} = \emptyset$. Not very interesting.

Now move to $\mathbb{P}^4$. If $\Gamma(\mathcal{A}) = P_3$, then $\bigcap \mathcal{A}$ must be a point.

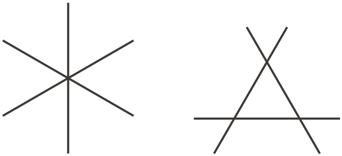


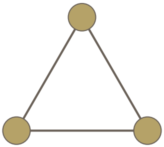
Proof.

$V_1 \cap V_2$ and $V_2 \cap V_3$ are lines in the projective plane V_2, and so they intersect in a common point. □

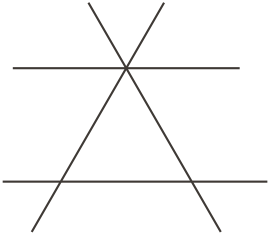
This actually works for all $n \geq 4$. If $\Gamma(\mathcal{A}) = P_3$, then $\operatorname{codim} \bigcap \mathcal{A} = 4$.

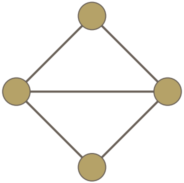
There are two types of arrangements with $\Gamma(\mathcal{A}) = K_3$, *starshaped* and *nonstarshaped*.





If $\Gamma(\mathcal{A})$ is a *diamond*, then one triangle must be starshaped and the other nonstarshaped.

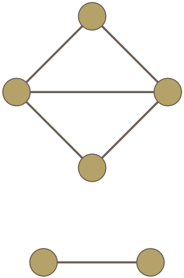




Definition

The *triangle graph* $T(G)$ of a graph G is the graph whose vertices are the triangle (K_3) subgraphs of G and whose edges are pairs of triangles which share a common edge.

If G is a diamond, then $T(G) = K_2$.



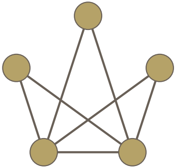
Theorem (–)

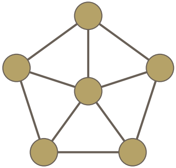
If G is K_4-free and $T(G)$ is not 2-colorable, then there are no subspace arrangements $\mathcal{A}$ with $\Gamma(\mathcal{A}) = G$.

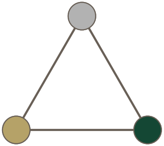
Proof.

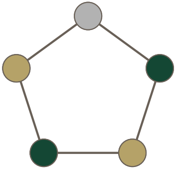
Each edge in $T(G)$ corresponds to a diamond in G. We color the corresponding vertices depending on whether each triangle is starshaped or nonstarshaped. □

Some examples of graphs which are *not* incidence graphs of subspace arrangements.

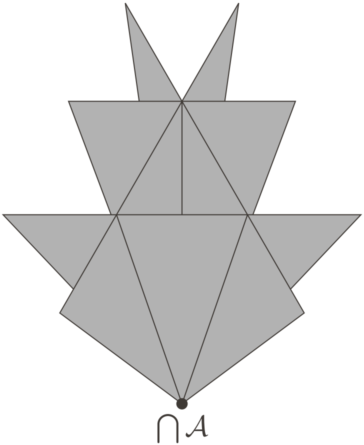








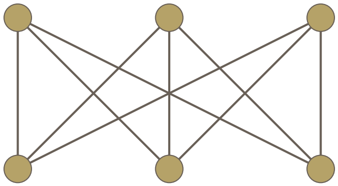
If $\Gamma(\mathcal{A})$ is a diamond, then $\operatorname{codim} \bigcap \mathcal{A} = 4$. (Same argument as P_3.)



What other families of graphs have this property?

Theorem (Teitler, –)

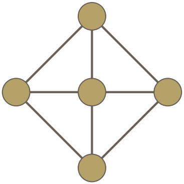
If $\Gamma(\mathcal{A})$ *is a complete bipartite graph, then* $\operatorname{codim} \bigcap \mathcal{A} = 4$*.*

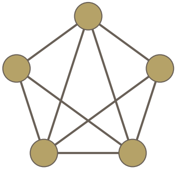


P_3 is a complete bipartite graph, but a diamond is not. Can we generalize the diamond?

Theorem (Nelson, –)

If $\Gamma(\mathcal{A})$ is a biconal graph, i.e., there are two vertices which are adjacent to all other vertices in the graph except each other, then $\operatorname{codim} \bigcap \mathcal{A} = 4$.





Thank you!