Enumeration of planar Tangles

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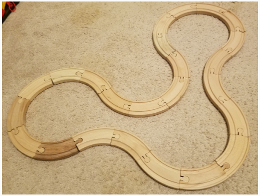
Tangles and circle packings

How many ways are there to build a toy train track with a given number of pieces?



Tangles

Simple case: the pieces are always quarter circles. This case coincides with another toy: the Tangle.

Length and class

Each piece of a Tangle is called a **link**. The number of links in a Tangle is its **length**.

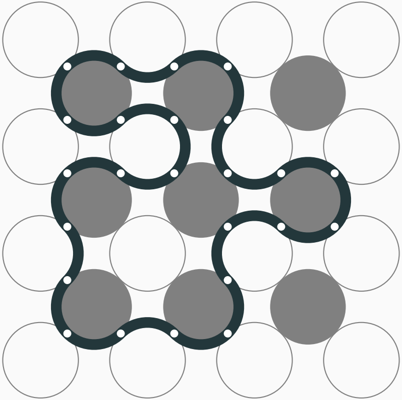
Theorem (Fleron)

The length of a Tangle is always a multiple of 4.

If a Tangle has length $4c$, then c is the **class** of the Tangle.

Square circle packing

The links of a Tangle necessarily belong to circles in a square packing of the plane.



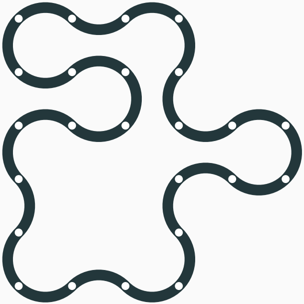
Coloring: black = interior, white = exterior

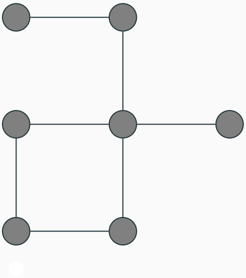
Dual graphs

Describing Tangles using graphs

The **dual graph** (actually a *polystick*) of a Tangle has:

- vertex set = black circles inside Tangle
- edge set = pairs of black circles that can be joined by a line segment without intersecting either the Tangle or any other circles in the packing





Relating class, size, and number of squares

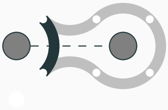
The **size** of a Tangle is the size (number of edges) of its dual graph.

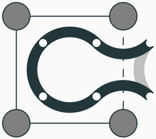
Theorem

If a Tangle of size m has k squares in its dual graph, then its class is

$$c = m - 2k + 1.$$

Proof: Induction on m and k.





Relating class and size

A polystick with m edges has

$$k \leq \frac{m + 1 - \sqrt{2m + 1}}{2}$$

squares (Buchholz/de Launey).

Corollary

If a Tangle has class c and size m, then

$$c - 1 \leq m \leq \frac{c^2 - 1}{2}.$$

Area

Theorem

If the links of a Tangle of size m belong to circles of radius r, then the area enclosed by the Tangle is $(4m + \pi)r^2$.

Proof. The vertices, edges, and squares of the dual graph all correspond to regions with known areas. The dual graph is planar, and so we may apply Euler's polyhedral formula.

Enumeration algorithm

How many Tangles are there?

One algorithm: enumerate polysticks (Redelmeier/Malkis), then throw out the ones with chordless cycles that aren't squares (Paton).

Fixed Tangles

m \ c	1	2	3	4	5	6	7	8	9	10	11
0	1										
1		2									
2			6								
3				22							
4			1		87						
5				8		364					
6					52		1574				
7				2		304		6986			
8					22		1706		31581		
9						182		9312		144880	
10					6		1288		50056		672390

One-sided Tangles

m \ c	1	2	3	4	5	6	7	8	9	10	11
0	1										
1		1									
2			2								
3				7							
4			1		24						
5				2		97					
6					14		401				
7				1		76		1772			
8					6		432		7930		
9						49		2328		36335	
10					2		326		12534		168249

Free Tangles

m \ c	1	2	3	4	5	6	7	8	9	10	11
0	1										
1		1									
2			2								
3				5							
4			1		15						
5				1		54					
6					9		212				
7				1		38		908			
8					4		224		4011		
9						28		1164		18260	
10					2		170		6299		84320

Polyominoes and growth constants

Growth constants?

Suppose

- $a_0(m) = \#$ fixed Tangles of size m
- $a_1(m) = \#$ one-sided Tangles of size m
- $a_2(m) = \#$ free Tangles of size m
- $\ell_0(c) = \#$ fixed Tangles of class c
- $\ell_1(c) = \#$ one-sided Tangles of class c
- $\ell_2(c) = \#$ free Tangles of class c

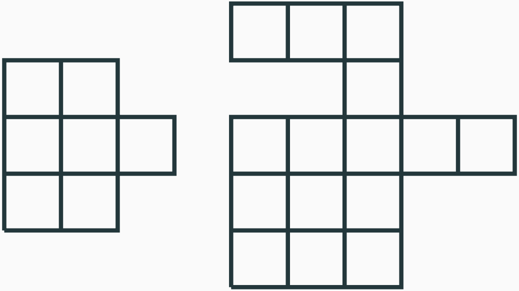
Do $\lim_{m\to\infty} a_i(m)^{1/m}$ and $\lim_{c\to\infty} \ell_i(c)^{1/c}$ exist? What are they?

Observations

- By gluing together Tangles and using log-superadditivity, these growth constants exist. (Klarner/Fekete)
- By the sandwich theorem, these growth constants are the same for $i = 0, 1, 2$.

Polyominoes

There are two families of polyominoes that have been associated with Tangles.



Chan (left) and Fleron (right) polyominoes

Polyominoes $\to$ bounds

Let $a_p(m)$ be the number of fixed hole-free polyominoes with m cells. Then

$$a_p(m+1) \leq a_0(m) \leq a_p(2m+1).$$

Let $\ell_p(c)$ be the number of fixed hole-free polyominoes with perimeter $2c$. Then

$$\ell_p(c+1) \leq \ell_0(c) \leq \ell_p(2c).$$

- lower bounds – Chan polyominoes
- upper bounds – Fleron polyominoes

Bounds on growth constants

Polyomino growth constants have been well-studied.

- $\kappa_p \approx 3.9709$ (hole-free polyominoes by area)
- $\mu_p \approx 2.6382$ (hole-free polyominoes by perimeter)

Theorem

Let $\kappa = \lim_{m\to\infty} a_i(m)^{1/m}$ *and* $\mu = \lim_{c\to\infty} \ell_i(c)^{1/c}$. *Then*

$$\kappa_p \leq \kappa \leq \kappa_p^2$$
$$\mu_p \leq \mu \leq \mu_p^2.$$

THANK YOU

Tangle font: http://erikdemaine.org/fonts/tangle/