Semiregular Tangles

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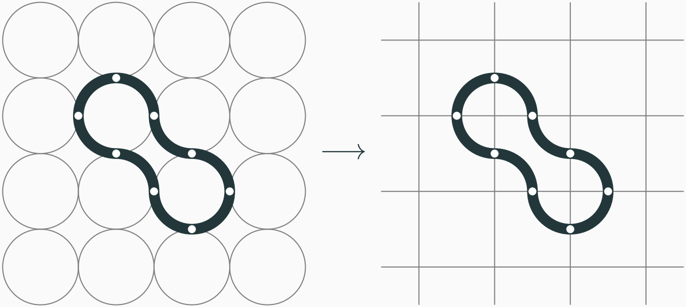
Tangles

A **Tangle** (named after the popular fidget toy) is a smooth closed plane curve made by gluing together arcs of circles with a common radius.

Most of the existing literature is devoted to **square Tangles**, in which the arcs are all quarter circles like the toys.

Observation

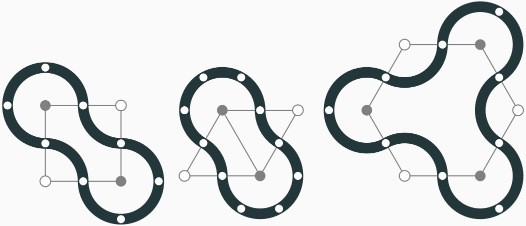
The arcs in a square Tangle all belong to circles in a *square circle packing*, which corresponds to the *square tiling* of the plane.



Regular Tangles

There are two other *regular* tilings of the plane: by equilateral triangles and regular hexagons that can be used to construct tangles with $\frac{1}{6}$-circle and $\frac{1}{3}$-circle links, respectively.

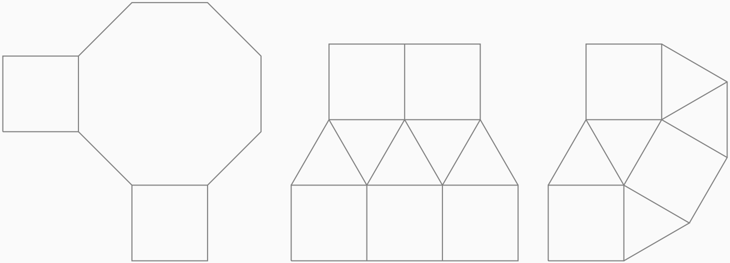
Each Tangle corresponds to a *dual polyform*, a set of polygons from the underlying tiling with a vertex coloring of its boundary.



Semiregular tilings

There are also eight additional *semiregular* tilings of the plane that use two or more regular polygons. We can use these to create new types of Tangles!

We focus on three of them: the *truncated square*, *elongated triangular*, and *snub square* tilings.



Question

For square Tangles, the *length* (i.e., the number of links) is always a multiple of four.

What can we say about the length of a semiregular Tangle?

Generalizing the notion of length

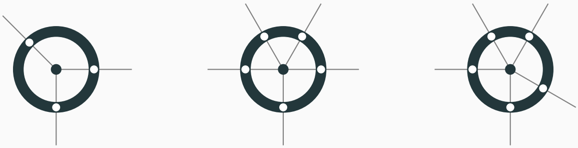
The *length* of a Tangle is the k-tuple $(\ell_1, \ldots, \ell_k)$, where

- k is the number of distinct polygons in the underlying tiling (so $k = 2$ in our cases),
- ℓ_i is the number of links that pass through a polygon of type i in the underlying tiling, and
- types are ordered so that if $i < j$, then a polygon of type i has fewer sides than a polygon of type j.

Length of a circle

The simplest Tangle is a circle, which corresponds to a single vertex in the tiling.

As a truncated square, elongated triangular, and snub square Tangle, the circle has length $(1, 2)$, $(3, 2)$, and $(3, 2)$, respectively.



Induction

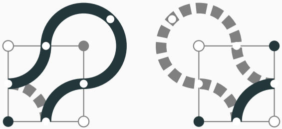
To prove our results about the length of semiregular Tangles, we use induction on the number of polygons in the dual polyform.

- *Base case:* What is the length of a circle?
- *Inductive step:* How does adding a small set of polygons to the dual polyform change the length of the Tangle?

For each type of Tangle, we will need to define a set of *fundamental operations* to use in this inductive step.

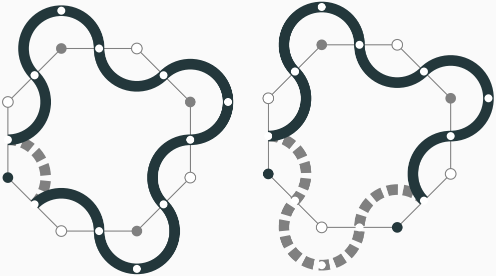
If there exists a sequence of fundamental operations taking a circle to a given Tangle, then the Tangle is *constructible*.

Truncated square Tangle fundamental operations – adding a square



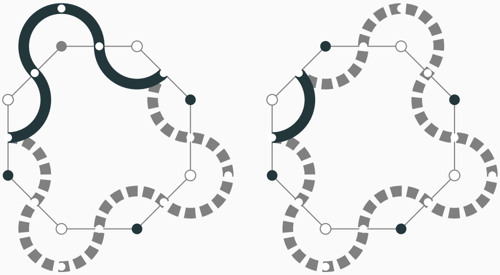
Net change in length: $(+1, +2)$ and $(-1, -2)$.

Truncated square Tangle fundamental operations – adding an octagon (part I)



Net change in length: $(+3, +6)$ and $(+1, +2)$.

Truncated square Tangle fundamental operations – adding an octagon (part II)



Net change in length: $(-1,-2)$ and $(-3,-6)$.

Truncated square Tangle length

Lemma

A constructible truncated square Tangle has length $(n, 2n)$ for some positive integer n.

Lemma

Every truncated square Tangle is constructible.

Theorem (Bale, Grant)

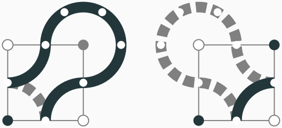
A truncated square Tangle has length $(n, 2n)$ for some positive integer n. Furthermore, a truncated square Tangle of length $(n, 2n)$ exists for all $n \neq 3$.

Elongated triangular and snub square Tangles

When triangles are in the tiling, things get trickier!

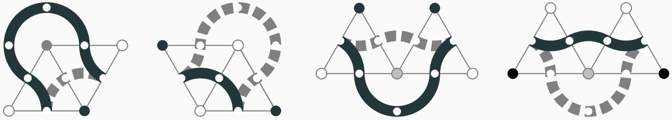
- Triangles aren't 2-colorable, so we need to add them in pairs.
- It's harder to identify fundamental operations.
- We don't know if every Tangle is constructible.

Elongated triangular Tangle fundamental operations – adding a square



Net change in length: $(+3, +2)$ and $(-3, -2)$.

Elongated triangular Tangle fundamental operations – adding pairs of triangles



Net change in length: $(+1, +2)$, $(-1, -2)$, $(-1, +2)$, and $(+1, -2)$.

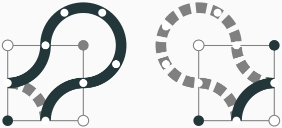
Elongated triangular Tangle length

Theorem (Madera)

A constructible elongated triangular Tangle has length $(m, 2n)$ *for positive integers* m *and* n.

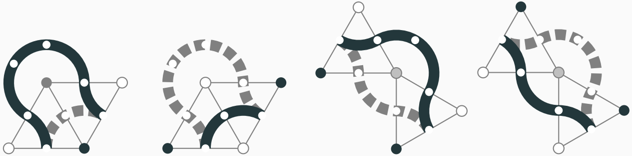
Are all elongated triangular Tangles constructible using these six fundamental operations?

Snub square Tangle fundamental operations – adding a square



Net change in length: $(+3, +2)$ and $(-3, -2)$.

Snub square Tangle fundamental operations – adding pairs of triangles



Net change in length: $(+1, +2)$, $(-1, -2)$, $(+1, 0)$, and $(-1, 0)$.

Elongated triangular Tangle length

Theorem (Boyce)

A constructible elongated triangular Tangle has length $(m, 2n)$ *for positive integers* m *and* n.

We may need to add some fundamental operations involving disjoint triangles.

THANK YOU

Tangle font: `http://erikdemaine.org/fonts/tangle/`