The Chow-Waring problem

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Polynomial decompositions

A simple example

Consider the binary quadratic form $f = x^2 + y^2 \in \mathbb{C}[x, y]$.

- f is a sum of two squares of linear forms
- $f = (x + iy)(x - iy)$, i.e., f is the product of two linear forms

Waring rank

Consider an $(n+1)$-ary d-ic f (i.e., $n+1$ variable, degree d).
What is the smallest s such that

$$f = \ell_1^d + \cdots + \ell_s^d$$

for linear forms ℓ_i? We call s the **Waring rank** of f.

For example, the Waring rank of $x^2 + y^2$ is 2.

Chow rank

Consider an $(n+1)$-ary d-ic f. What is the smallest s such that

$$f = \ell_{1,1} \cdots \ell_{1,d} + \cdots + \ell_{s,1} \cdots \ell_{s,d}$$

for linear forms $\ell_{i,j}$? We call s the **Chow rank** of f.

For example, the Chow rank of $x^2 + y^2$ is 1.

Chow-Waring rank

Let $\mathbf{d} = (d_1, \ldots, d_k)$ with $d_1 + \cdots + d_k = d$.

Consider $(n+1)$-ary d-ic f. What is the smallest s such that

$$f = \ell_{1,1}^{d_1} \cdots \ell_{1,k}^{d_k} + \cdots + \ell_{s,1}^{d_1} \cdots \ell_{s,k}^{d_k}$$

for linear forms $\ell_{i,j}$? We call s the **Chow-Waring rank** of f for $\mathbf{d}$.

- $\mathbf{d} = (d) \implies$ Waring rank
- $\mathbf{d} = (1, \ldots, 1) \implies$ Chow rank

Secant varieties

Veronese varieties

Every $(n+1)$-ary linear form corresponds to a point in $\mathbb{P}^n$.

Every $(n+1)$-ary d-ic corresponds to a point in $\mathbb{P}^{\binom{n+d}{d}-1}$.

We define the **Veronese map**

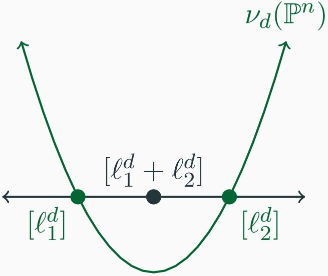
$$\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$$

$$[\ell] \mapsto [\ell^d]$$

The image $\nu_d(\mathbb{P}^n)$ is known as a **Veronese variety**.

Secant lines

Forms of Waring rank 2 lie on a *secant line* to a Veronese variety.



Forms of Waring rank s lie on a *secant* $(s-1)$*-plane* to a Veronese variety. In particular, they lie on the sth *secant variety* $\sigma_s(\nu_d(\mathbb{P}^n))$:

$$\sigma_s(X) = \overline{\bigcup_{p_1,\ldots,p_s \in X} \langle p_1, \ldots, p_s \rangle}$$

Waring rank of generic forms

If $\sigma_s(\nu_d(\mathbb{P}^n)) = \mathbb{P}^{\binom{n+d}{d}-1}$, then (almost) all $(n+1)$-ary d-ics have Waring rank $\leq s$.

Goal: Find the smallest s such that

$$\dim \sigma_s(\nu_d(\mathbb{P}^n)) = \binom{n+d}{d} - 1.$$

Chow-Waring rank of generic forms

More generally, the Chow-Waring rank of a generic form for $\mathbf{d} = (d_1, \ldots, d_k)$ is the smallest s such that

$$\dim \sigma_s(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n)) = \binom{n+d}{d} - 1,$$

where the **Chow-Veronese variety** $\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n)$ is the image of the map

$$(\mathbb{P}^n)^{\times k} \to \mathbb{P}^{\binom{n+d}{d}-1}$$

$$([\ell_1], \ldots, [\ell_k]) \mapsto [\ell_1^{d_1} \cdots \ell_k^{d_k}]$$

Dimensions of secant varieties

Expected dimension

Based on a naïve parameter count, we see that for $X \subset \mathbb{P}^{\binom{n+d}{d}-1}$,

$$\dim \sigma_s(X) \leq s \dim X + s - 1 = s(\dim X + 1) - 1.$$

So:

$$\dim \sigma_s(X) \leq \min\left\{ s(\dim X + 1), \binom{n+d}{d} \right\} - 1.$$

We call this upper bound the **expected dimension** of $\sigma_s(X)$, denoted $\operatorname{expdim} \sigma_s(X)$.

Expected Chow-Waring rank

Since $\dim \mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n) = kn$, where $\mathbf{d} = (d_1, \ldots, d_k)$, we see that if s is the Chow-Waring rank of a generic $(n+1)$-ary d-ic for $\mathbf{d}$ and if $\sigma_s(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n)$ has the expected dimension, then s is the smallest integer for which

$$s(kn+1) \geq \binom{n+d}{d},$$

and so

$$s = \left\lceil \frac{1}{kn+1} \binom{n+d}{d} \right\rceil .$$

Question: When does this work?

Defective cases

Cases for which $\dim \sigma_s(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n))) < \operatorname{expdim} \sigma_s(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n))$:

- $\mathbf{d} = (2)$, $n \geq 2$, $2 \leq s \leq n$
- $\mathbf{d} = (1,1)$, $n \geq 4$, $2 \leq n \leq \frac{n}{2}$
- $\mathbf{d} = (3)$, $n = 4$, $s = 7$
- $\mathbf{d} = (2,1)$, $2 \leq n \leq 4$, $s = n$
- $\mathbf{d} = (4)$, $2 \leq n \leq 4$, $s = \binom{n+2}{2} - 1$

Conjecture: This is it!

Question

For the other cases, how can we show that

$$\dim \sigma_s(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n))) = \operatorname{expdim} \sigma_s(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n))?$$

Techniques

Lemma (Terracini)

Let $p_1, \ldots, p_s \in X$ *be generic. Then*

$$\dim \sigma_s(X) = \dim\langle T_{p_1}X, \ldots, T_{p_s}X\rangle.$$

We can reduce the problem of finding the dimension of a secant variety to finding the rank of a matrix!

For $X = \mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n)$, these tangent spaces are straightforward to construct using the product rule from calculus.

Idea: Choose the points $p_1, \ldots, p_s$ carefully so we can use induction and semicontinuity.

Abundancy

Fact

(subabundant) Let $s_1 = \left\lfloor \frac{1}{kn+1}\binom{n+d}{d} \right\rfloor$.

$$\dim \sigma_{s_1}(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n)) = s_1(kn+1) - 1 \implies$$
$$\forall s \leq s_1, \dim \sigma_s(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n)) = s(kn+1) - 1$$

(superabundant) Let $s_2 = \left\lceil \frac{1}{kn+1}\binom{n+d}{d} \right\rceil$.

$$\dim \sigma_{s_2}(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n)) = \binom{n+d}{d} - 1 \implies$$
$$\forall s \geq s_2, \dim \sigma_s(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n)) = \binom{n+d}{d} - 1$$

Only need two cases for each pair $n, \mathbf{d}$

If

$$\dim \sigma_{s_i}(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n)) = \operatorname{expdim} \sigma_{s_i}(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n))$$

for $i = 1, 2$, then

$$\dim \sigma_s(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n)) = \operatorname{expdim} \sigma_s(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n))$$

for all $s \in \mathbb{N}$.

Quasipolynomials

Fact

$\left\lfloor \frac{1}{kn+1}\binom{n+d}{d} \right\rfloor$ and $\left\lceil \frac{1}{kn+1}\binom{n+d}{d} \right\rceil$ are **quasipolynomial** functions of degree $d-1$ for $n \gg 0$, i.e., there exists some ℓ and degree $d-1$ polynomial functions $s_0, \ldots, s_{\ell-1}$ such that

$$\left\lfloor \frac{1}{kn+1}\binom{n+d}{d} \right\rfloor = s_r(n) \text{ if } n \equiv r \pmod{\ell}$$
$$\left\lceil \frac{1}{kn+1}\binom{n+d}{d} \right\rceil = s_r(n) + 1 \text{ if } n \equiv r \pmod{\ell}$$

Backward difference operator

Suppose s is a polynomial function of degree $d - 1$. Then the **backward difference operator** ∇ with step size ℓ is:

$$\begin{aligned} \nabla^0 s(n) &= s(n) \\ \nabla^i s(n) &= \nabla^{i-1} s(n) - \nabla^{i-1} s(n - \ell) \quad \forall i \geq 1 \end{aligned}$$

In particular, $\nabla^d s(n) = 0$.

Newton backward difference formula

Theorem

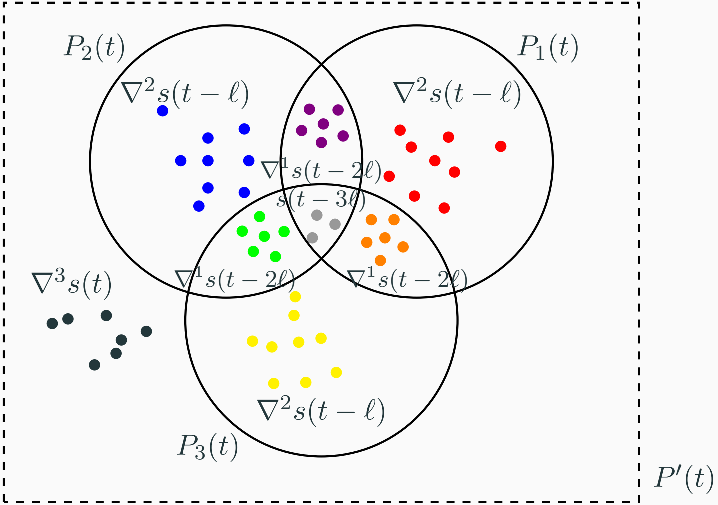
$$s(n) = \sum_{j=0}^{d} \binom{d}{j} \nabla^{d-j} s(n - j\ell).$$

Observation: Suppose we have d linear subspaces of $\mathbb{P}^{\binom{n+d}{d}-1}$. $\binom{d}{j}$ is also the number of ways in which j of these subspaces can intersect.

Brambilla-Ottaviani lattice

Choose $s(n)$ points on $\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n)$ so that each element of the intersect lattice of d subspaces gets the number of points corresponding to a summand of the Newton backward difference formula.

Brambilla-Ottaviani lattice (visualized)



Induction

If we compute the the ranks of matrices constructed using Terracini's lemma for $n \leq d\ell + 1$ as base cases and get the expected value, then the dimension is the expected one for *all* n.

If we have the expected dimension for these specialized cases, then we have the expected dimension for the general case by semicontinuity.

Results for $d = 3$

Using this technique:

Theorem

Except for the known defective cases, $\sigma_s(\mathrm{CV}_{\mathbf{d}}(\mathbb{P}^n))$ has the expected dimension for

- $\mathbf{d} = (3)$ *[Brambilla, Ottaviani (2008)]*[1] *($\ell = 3$)*
- $\mathbf{d} = (2, 1)$ *[Abo, Vannieuwenhoven (2018)] ($\ell = 24$)*

[1] Previously proven, using different techniques, in [Alexander, Hirschowitz (1995)] and [Chandler (2002)]

Results for the Chow variety

Fact

Since $\frac{1}{dn+1}\binom{n+d}{d}$ is symmetric in n and d, this technique also works in the Chow case ($\mathbf{d} = (1, \ldots, 1)$) when fixing n and using induction on d

Theorem

$\sigma_s(\mathrm{CV}_{(1,\ldots,1)}(\mathbb{P}^n))$ *has the expected dimension for*

- $n = 2$ *[Abo (2014)] (*$\ell = 4$*)*
- $n = 3$*,* $d = 3$ *[Abo (2014)] (partial,* $\ell = 6$*)*
- $n = 3$ *[T. (2013)] (partial,* $\ell = 9$*)*
- $n = 3$*,* $d = 3$ *[T., Vannieuwenhoven (2021)] (complete,* $\ell = 27$*)*

Horace differential lemma

Horace $\implies$ cubics as base case

Theorem

- *If* $\sigma_s(\mathrm{CV}_{(3)}(\mathbb{P}^n))$ *has the expected dimension, then* $\sigma_s(\mathrm{CV}_{(d)}(\mathbb{P}^n))$ *has the expected dimension for* $d \geq 3$. *[Alexander, Hirschowitz (1992)]*
- *If* $\sigma_s(\mathrm{CV}_{(2,1)}(\mathbb{P}^n))$ *has the expected dimension, then* $\sigma_s(\mathrm{CV}_{(d-1,1)}(\mathbb{P}^n))$ *has the expected dimension for* $d \geq 3$. *[Bernardi, Catalisano, Gimigliano, Idá (2009)]*

Question: Can we do this for other $\mathbf{d}$?

Another approach

Idea: Apply a technique from [Abo, Ottaviani, Peterson (2009)] on the secant varieties of Segre varieties.

Suppose A is the matrix whose rank determines the dimension of $\sigma_s(\mathrm{CV}_{(1,\dots,1)}(\mathbb{P}^n))$. By careful choice of our points $p_1, \dots, p_s$, we can find matrices B, C, and D corresponding to spaces of forms with n variables and degrees d, $d-1$, and $d-2$, respectively, such that

$$A = \begin{pmatrix} B & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & D \end{pmatrix}.$$

Then

$$\operatorname{rank} A = \operatorname{rank} B + \operatorname{rank} C + \operatorname{rank} D.$$

Induction on n

Using induction, we obtain the following result.

Theorem (T. (2017))

If

$$\dim \sigma_s(\mathrm{CV}_{(1,\dots,1)}(\mathbb{P}^{n_0})) = s(dn_0 + 1) - 1,$$

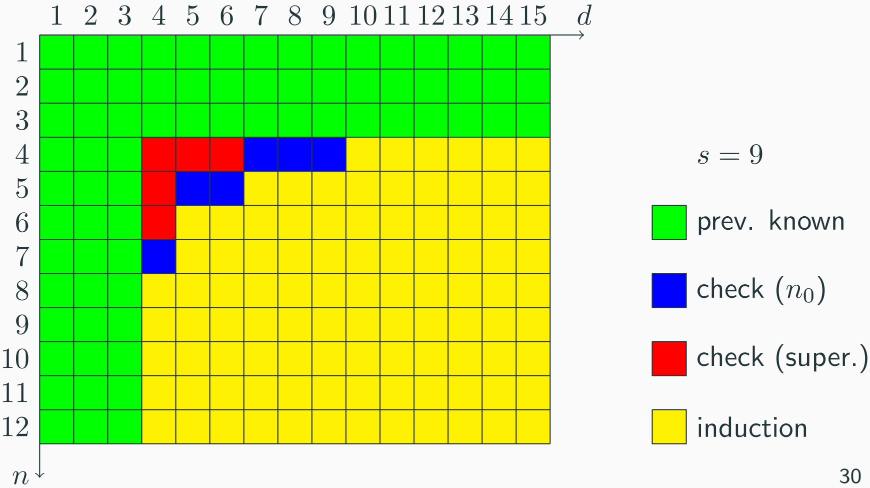
then

$$\dim \sigma_s(\mathrm{CV}_{(1,\dots,1)}(\mathbb{P}^{n})) = s(dn + 1) - 1$$

for all $n \geq n_0$.

Fixed s

For fixed s, this reduces finding $\dim \sigma_s(\mathrm{CV}_{(1,\ldots,1)}\,\mathbb{P}^n))$ for all n, d to checking finitely many base cases.



Nondefective for small s

Using Macaulay2 to check as many of these base cases as possible, we obtain the following result.

Theorem (T. (2017))

If $s \leq 35$, then $\sigma_s(\mathrm{CV}_d(\mathbb{P}^n))$ has the expected dimension except for the previously known defective cases.

Thank you!